PROPERTIES OF POINT MASS LENSES ON A REGULAR POLYGON AND THE PROBLEM OF MAXIMUM NUMBER OF IMAGES

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We study the critical curves, caustics, and multiple imaging due to one of the simplest many-body gravitational lens configurations: equal-mass point masses on the vertices of a regular polygon. Some examples of the critical curves and caustics are also displayed. We pose the problem of determining the maximum number of lensed images due to regular-polygon and general point mass configurations. Our numerical simulations suggest a maximum that is linear, rather than quadratic, in the number of point masses.

1 Critical Curves and Caustics

Suppose that g point masses m_k, where $m_k = 1/g$, are on the vertices z_k of a radius r regular polygon centered at the origin: $z_k = r\ e^{i2\pi(k-1)/g}$, where $k = 1, \ldots, k$ and $g \geq 2$. [A point mass lens ($g = 1$) produces a circular critical curve and point caustic; light sources off the point caustic have two lensed images.] The associated *lens equation*, expressed in complex quantities (Bourassa, Kantowski & Norton, [1] Witt [2]), is given by

$$z_s = z - \sum_{k=1}^{g} \frac{m_k}{\bar{z} - \bar{z}_k} = z - \frac{\bar{z}^{g-1}}{\bar{z}^g - r^g}, \tag{1}$$

where z_s is the light-source position. The lens equation defines a mapping, $\eta : z \mapsto z_s$, from the lens plane into the light source plane. *Lensed images* of a light source at z_s are solutions z in $\mathbf{C}$ of the lens equation. *Critical curves* (i.e., set of all infinitely magnified lensed images) are given by setting the Jacobian determinant J of η equal to zero:

$$J = 1 - \frac{\partial z_s}{\partial \bar{z}} \overline{\frac{\partial z_s}{\partial \bar{z}}} = 0.$$

This is solved by $\frac{\partial z_s}{\partial \bar{z}} = e^{i\phi}$, where $0 \leq \phi < 2\pi$. The critical curves are then the solution curves $z(\phi)$, where $0 \leq \phi < 2\pi$, of

$$p(z) = z^{2g} + e^{i\phi} z^{2g-2} - 2r^g z^g + (g-1) r^g e^{i\phi} z^{g-2} + r^{2g} = 0. \tag{2}$$

Since $z = 0$ is not a root of Eq.(2), *no critical curve and no caustic passes through the origin. Caustics* (i.e., set of positions from which a light source has at least one

infinitely magnified lens image) are the η-images of critical curves. By Eq.(2), *there are at most* $2g$ *critical curves; hence, the same for caustics.* If critical curves merge for a given parameter value, then $p(z)$ has a double or higher order zero. But $\frac{\partial^2 z_s}{\partial \bar{z}^2}$ is equivalent to a complex polynomial in z of degrees 3 and 6 for $g = 2$ and 3,resp., and degree $2g+1$ for $g \geq 4$. In the latter case, one solution is $z = 0$, which cannot lie on no critical curve or caustic. Thus, *if* $g = 2, 3$, *then there are at most* $3, 6$ *beak-to-beak caustics, resp., while at most* $2g$ *occur for* $g \geq 4$. A general g-point mass system has at most $3g - 3$ beak-to-beaks (Witt & Petters [3]).

Examples of the critical curves and caustics are shown in Figure 1 as a function of r for the case $g = 6$.

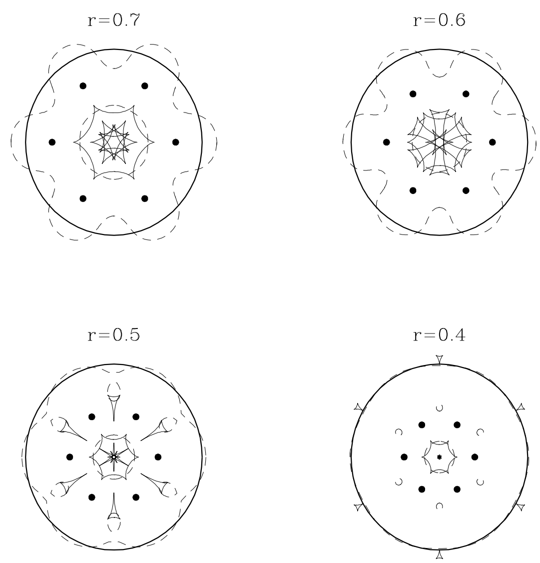

Figure 1: Critical curves (dashed lines) and caustics (thin solid lines) are shown for equal-mass sextuple lenses (filled dots) as a function of the radius of the regular polygon. A unit circle is indicated with the thick solid line.

2 Number of Lensed Images of a Light Source at the Center

For a light source at the center of the polygon ($z_s = 0$), the lens equation becomes

$$\rho = \frac{\rho^{g-1}}{\rho^g - r^g e^{ig\theta}}, \tag{3}$$

where we expressed z in the polar coordinates $\rho e^{i\theta}$. An immediate solution is $\rho = 0$. For any other solution, $e^{ig\theta}$ must be real and, hence, is either $+1$ or

-1. Eq.(3) can be recast into $f_\pm(\rho) \equiv \rho^g - \rho^{g-2} \pm r^g = 0$, where f_+ and f_- correspond to the lens equation with $e^{i\theta} = -1$ and $+1$, resp. If $n_\pm$ are the number of positive zeros of $f_\pm$, then the total number of images is then simply given by, $N = g(n_+ + n_-) + 1$, where the factor of g arises due to rotational symmetry. If $g = 2$, then $n_+ = 0, 1$ for $r \geq 1$ and $r < 1$, resp., and $n_- = 1$. *It follows that* $N = 3$ *or* 5 *if* $g = 2$. Now, suppose that $g \geq 3$. By Descartes' rule of signs, we have $n_\pm \leq$ #(sign changes in coefficients of $f_\pm$). Consequently, $n_+ \leq 2$ and $n_- \leq 1$ for $g \geq 3$. Hence, $N \leq 3g + 1$. This upper bound is also the maximum, that is, it is achievable for each g. In fact, let $r_{cr} = \left(\rho_m^{g-2} - \rho_m^g\right)^{1/g}$, where $\rho_m = [(g-2)/g]^{1/2}$. It can be shown that if $g \geq 3$, then $n_- = 1$ and $n_+ = 0, 1, 2$ for $r > r_{cr}$, $r = r_{cr}$, $r < r_{cr}$, resp. Thus, *if* $g \geq 3$, *then* $N = g+1, 2g+1, 3g+1$ *for* $r > r_{cr}$, $r = r_{cr}$, *and* $r < r_{cr}$, *resp.*

3 Open Problem

The bounds on the total number of lensed images due to g point masses (not necessarily on a regular polygon) are known to be $g + 1 \leq N \leq g^2 + 1$. The lower bound follows rigorously from Morse theory (Petters [4]), while the upper bounds can be shown using a trick substitution (Witt — see Ref. 2), or, via resultants (Petters [5]). The lower bound $g + 1$ is achievable for each g (Petters [6]); hence, it is the minimum number of lensed images. We do not know whether the upper bound $g^2 + 1$ is attainable for each g. For g point masses on the vertices of a regular polygon, and light source not necessarily at the origin, the maximum number of images appears to be $3g + 1$. Our numerical simulations for generic point-mass configurations seem to confirm this limit as well. *It is unknown to the authors whether the maximum number of lensed images due to a general g point mass lens system is linear, i.e., does $N_{max} = gn_1 + n_2$ for each $g \geq 1$, where n_1 and n_2 are fixed positive integers?*

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